Comments

Solution of Large Numbers of Simultaneous Equations

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THERE has been considerable discussion in the industry of late concerning methods of solving large numbers of simultaneous equations. However, there has not been much said of actual sizes that have been solved and checked out successfully. This note is written to present the results of some experience with this problem and to encourage further discussion of actual experiences.

Vought Astronautics Division has been using the routine reported in Ref. 1 for the investigation of thermal stresses in thick spherical shells subjected to nonlinear temperature distributions. In this routine, Dr. A. J. A. Morgan and C. H. Christensen applied the suggestions of David Young[2, 3] (for using over-relaxation techniques applied to the Gauss-Seidel iteration process) to the solution of a set of linear difference equations. The solution of these equations yields the displacements of points on a finite difference net spread over the shell domain.

The routine was programmed for the IBM 704 DPS. It is being used with the 7090 system under a compatibility program which slows execution considerably compared to a pure 7090 programmed routine. The exact extent to which the routine is slowed is not known. For 211 equations, running time averages about 7 min; 410 equations, $\frac{1}{2}$ hr. The over-relaxation device accelerates convergence over pure iteration by somewhat over an order of magnitude for an average case. Excellent solutions were obtained as evidenced by checks against simple closed form solutions and extensive checks for consistency and static equilibrium.

The writer has been under the impression that the usual methods of matrix inversion tend to become unreliable at sizes somewhere in the neighborhood of 100×100 to 150×150 because of the accumulation of roundoff. This impression is based upon a very shaky foundation as it results from a small amount of actual experience, and it is recognized that much variation should be expected from problem to problem with different matrices and types of solution. It has been assumed that larger sizes would require some form of iteration to minimize this round-off problem. One proposal considered at Vought Astronautics Division involved adding a few Gauss-Seidel iteration passes to a normal matrix inversion. This should extend the range of normal techniques considerably.

Further discussion should be of value to the industry.

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Comment on the Use of Superposition for Plates under Combined Loading

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A RECENT series of papers[1–4] contains statements to the effect that plate problems involving combined loading (both midplane and bending loads) can be solved by the superposition of a pure bending solution and a plane stress solution. The authors, Newman and Forray, do not indicate clearly the assumptions made or the resulting restrictions upon the class of problems which can be solved by this superposition. References 1–3 consider plates subjected to loads that are both mechanical and thermal in nature. The use of the superposition is justified by the use of "classical linear plate" theory and, in Ref. 3, by the additional words "linear thermal stress" theory. The position of Newman and Forray on this matter is stated most clearly in Ref. 4, where the introductory remarks are

> One of the basic assumptions of the classical linear theory of plates is that the bending action does not induce significant midplane stretching. It is further assumed that stresses and deformations produced by loads and restraints in the midplane can be superposed on the bending solution. Thus coupling between the two effects is not accommodated by the classical theory.

No one will argue about the first sentence, of course. The purpose of this note is to express some concern about the emphasis placed on this superposition principle and to inquire whether or not the proposed superposition has a useful region of validity.

Timoshenko[5] credits St. Venant with the derivation of the proper equation for the case of combined loading in plates. The equation was published in 1883 and generally is considered a part of classical linear plate theory. With the addition of a term due to thermal moments, it is of the form

$$D\nabla w^4 - N_x \frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - N_y \frac{\partial^2 w}{\partial y^2} = q - \frac{1}{1-\nu}\nabla^2 M_T \quad (1)$$

The symbols N_x,N_{xy}, and N_y represent the midplane forces per unit length and are due to midplane loading. These forces are not to be confused with the nonlinear midplane forces due to stretching of the plate out of its plane. This is a linear equation, valid for the small deflections of a plate made of an elastic material. It is well known that the solution of the combined loading problem is accomplished first by finding N_x,N_{xy}, and N_y from a plane stress solution and then by solving Eq. (1). If superposition of solutions is to be used, the argument must be that the restoring forces due to the membrane forces are much smaller than the bending forces throughout the plate:

$$\left| N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right| \ll |D\nabla w^4| \quad (2)$$

This assumption places a drastic restriction on the magnitude of the midplane forces. It is doubtful that the early authors intended that the physical problem of a plate subjected to midplane and transverse loads be treated under the assumption of inequality (2).

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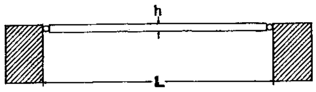

Fig. 1 Plate strip

The problem now will be considered in more detail. In Ref. 2, for instance, it is stated (in connection with a circular plate problem) that "For thin plates the temperature may be approximated by $T_0(r,\theta,t) + zT_1(r,\theta,t)$ where the effects of the two terms may be superimposed if the deflections are small and the material is linearly elastic." Newman and Forray place no restriction on the magnitude of T_0 or the corresponding forces caused by this term. The assumption of inequality (2) is needed, however. An example now will be given in order to indicate how restrictive this inequality can be. Consider a "two-dimensional" plate strip for simplicity (Fig. 1). The plate is pinned between two immovable supports and is of length L, thickness h, and made of aluminum. It is loaded by a uniform cooling of $T_0°$ and by a system of thermal moments. For a plate 28 in. long and 0.100 in. thick which deflects in an approximate half-sine wave, it is found that the superposition principle will give an answer in error by 100% if T_0 is as large as 1°F. In other words, neglecting the midplane restoring forces due to a 1° temperature drop will cause the calculated deflection to be twice as large as the correct value. For an accurate answer, T_0 would have to be limited to much less than 1°F in this case! (It is realized that a temperature rise of only 1°F would cause this same plate to buckle. This is a different question, however, and is not of interest here.)

Of course, counter-examples can be constructed which would be less restrictive, particularly for thicker plates and for plates free to expand. For thin plates in general, however, it appears that the superposition is applicable only to cases where the midplane stresses are so low that they may as well be neglected in the determination of the maximum stresses. If this is the case, then the superposition is not needed.

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Reply by Authors to William J. Anderson

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ANDERSON states in his note that the inclusion of membrane restoring forces in the plate equilibrium equation is "generally considered a part of classical linear plate theory." This statement is debatable, since Eq. (1) cannot be derived from variational principles without the inclusion of nonlinear terms in the strain-displacement relations. Novozhilov[1] remarks that when these nonlinear terms are neglected "one obtains the formulas of the classical theory of plates." Furthermore, regardless of what the expression "classical plate theory" means, the present authors clearly stated, in the references cited by Anderson, that superposition of the bending and in-plane problems is being used. It is true that the limitations of this superposition technique have not been established. However, the determination of the range of applicability of this method often presents a very difficult task. In the class of problems investigated, the temperature and hence the membrane forces are permitted to vary over the plate planform. If the deflections are small, Anderson's requirement that the inequality (2) be satisfied *throughout* the plate is a sufficient but not necessary condition for the applicability of superposition. It is far too restrictive, since superposition still may be used when (2) is satisfied on some average basis rather than pointwise. For example, superposition should yield accurate solutions (with interior bending and membrane stresses of the *same* order of magnitude) for a wide variety of nonuniform heating problems in which the edges are unrestrained in the plane of the plate. These are very practical problems, since important structural components usually are designed to permit thermal expansion. As a guide to the reader, superposition generally can be considered valid when the absolute ratio of the generated edge thrust to the buckling thrust is small compared to unity.

No one will deny the inaccuracy of superposition for an axially restrained thin strip. The authors never intended, however, that the technique be used indiscriminately.

In closing, it must be remarked that Refs. 2–8 present solutions to many thermoelastic in-plane and bending problems. Anderson neglects to mention that either type of problem is of importance in itself.

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Comment on "Error Matrix for a Flight on a Circular Orbit"

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IN a recent technical note,[1] Wisneski derives the well-known solutions of the linear perturbation differential

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